2015年　電磁気 A

$\boxed{1}$ (1)

$$\vec{\nabla} f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \qquad \vec{\nabla}^2 f = 2+2+2 = 6$$

(2)

$$\mathrm{div}\,\vec{A} = 0+0+1 = 1$$

$$\mathrm{rot}\,\vec{A} = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

(3) (a) $f = r = \sqrt{x^2+y^2+z^2}$

$$(\mathrm{rot}\,\vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = -\frac{y}{r^2}\cdot\frac{x}{r} + \frac{x}{r^2}\cdot\frac{y}{r} = 0$$ 他も同じ

(b) $\mathrm{rot}\,\vec{A} \neq 0$ なので 存在しない

(4) (b)

$$\mathrm{rot}\,(\mathrm{grad}\,f)_z = \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = 0$$

(c)

$$\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 0$$

(5) (a)

$$\int_S \vec{A}\cdot d\vec{S} = \int_V (\mathrm{div}\,\vec{A})\cdot dV$$

($\vec{A}$：ベクトル場，S：閉曲面，V = Sで囲まれる領域)

(b)

$$\oint_C \vec{A}\cdot d\vec{r} = \int_S \mathrm{rot}\,\vec{A}\, d\vec{S}$$

($\vec{A}$：ベクトル場，Cが閉経路，SがCに囲まれる曲面)

(6)

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \qquad \begin{cases} dx = -\sin\theta\, d\theta \\ dy = \cos\theta\, d\theta \end{cases}$$

$$\int_C \vec{A}\cdot d\vec{r} = \int_C \begin{pmatrix} -y/r \\ x/r \\ 0 \end{pmatrix}\cdot\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$= \int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta$$

$$= \int_0^{2\pi} 1\, d\theta = 2\pi$$

(7) ファラデーの法則

$$\mathrm{rot}\,\vec{E} = -\frac{\partial B}{\partial t}$$

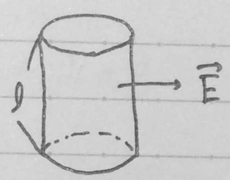
$$\int_S \mathrm{rot}\,\vec{E}\, d\vec{S} = \int_S -\frac{\partial B}{\partial t}\, d\vec{S}$$

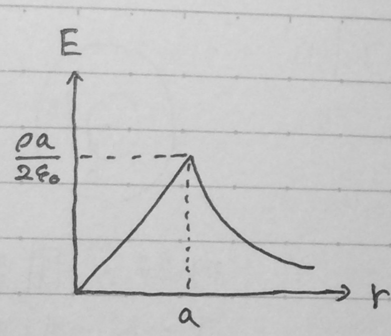
ストークスの定理より

$$\oint \vec{E}\cdot d\vec{r} = \int_S -\frac{\partial B}{\partial t}\, d\vec{S}$$

∴ 経路を一周したときの起電力が $\int_S -\frac{\partial B}{\partial t} d\vec{S} = -\frac{d\Phi}{dt}$ となる。

$\boxed{2}$ (1) $|\vec{E}| \cdot l \cdot 2\pi r = \dfrac{\pi r^2 l \cdot \rho}{\varepsilon_0}$

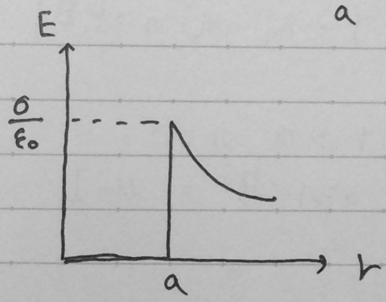




$$|\vec{E}| = \frac{\rho r}{2\varepsilon_0} \quad (r < a)$$
$$|\vec{E}| = \frac{a^2 \rho}{2r\varepsilon_0} \quad (r > a)$$

(2) $|\vec{E}| \cdot 2\pi r l = 2\pi l a \sigma / \varepsilon_0$

$$\therefore |\vec{E}| = \frac{a\sigma}{r\varepsilon_0} \quad (r > a)$$
$$|\vec{E}| = 0 \quad (r < a)$$



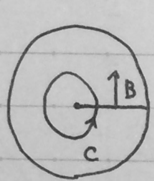
(3) 全微分公式より

$$-\int_c \vec{E} \cdot d\vec{r} = +\int_c \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$
$$= \int_c d\phi = \phi(\vec{r_2}) - \phi(\vec{r_1})$$

(4) $\dfrac{Q}{\varepsilon_0} = E \cdot 2\pi r \cdot L$

$$\therefore E = \frac{Q}{2\pi r L \varepsilon_0}$$
$$\int_a^b E\, dr = \int_a^b \frac{Q}{2\pi r L \varepsilon_0} dr$$
$$V = \frac{Q}{\varepsilon_0} \cdot \frac{1}{2\pi L} \cdot \log \frac{b}{a}$$
$$\therefore Q = \frac{2\pi L \varepsilon_0}{\log \frac{b}{a}} V$$
$$\therefore C = \frac{2\pi L \varepsilon_0}{\log \frac{b}{a}}$$

$\boxed{3}$ (1)

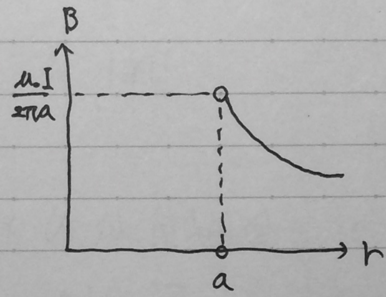


Bの向きは円筒の接線の方向

内部の閉経路Cをとると、Cの内部は電流はないので
アンペールの法則より　$B = 0$

(2)　$r > a$ のとき

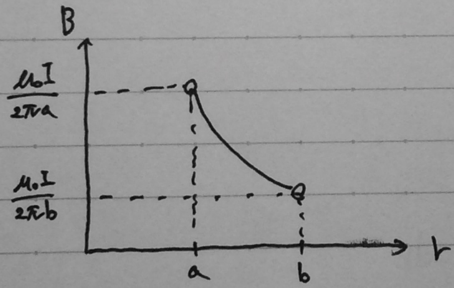
$2\pi r B = \mu_0 I$　　　$\therefore B = \dfrac{\mu_0 I}{2\pi r}$



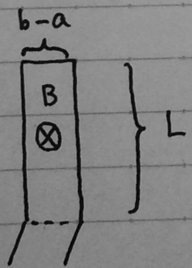
(3)　$a < r < b$ のとき

$2\pi r B = \mu_0 I$ のとき　　$B = \dfrac{\mu_0 I}{2\pi r}$

$r > b$ のとき　　$B = 0$



(4)



$$-L_I \frac{dI}{dt} = V = -\frac{d\Phi}{dt}$$

$$\therefore \Phi = I \cdot L_I$$

$$\int_a^b \frac{\mu_0 I}{2\pi r} \cdot L \cdot dr = I L_I$$

$$\therefore L_I = \frac{\mu_0}{2\pi} \cdot L \cdot \log \frac{b}{a}$$

$\boxed{4}$ (1) $\frac{Q}{C} + L\frac{dI}{dt} = 0$

$Q = -CL\ddot{Q}$

$\therefore Q = A\cos\frac{1}{\sqrt{CL}}t$

$t=0$のとき $Q_0 = A\cdot 1$

$\therefore Q = Q_0 \cos\frac{1}{\sqrt{CL}}t$

$\therefore I = -Q_0\frac{1}{\sqrt{CL}}\sin\frac{1}{\sqrt{CL}}t$

(2) $\begin{cases}\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 V}{\partial y\partial x}\\ \frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 V}{\partial y^2}\end{cases}$ $\begin{cases}\frac{\partial^2 u}{\partial y\partial x} = -\frac{\partial^2 V}{\partial x^2}\\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 V}{\partial x\partial y}\end{cases}$

ここで $\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$, $\frac{\partial^2 V}{\partial x\partial y} = \frac{\partial^2 V}{\partial y\partial x}$

より、 $\frac{\partial^2 V}{\partial y^2} = -\frac{\partial^2 V}{\partial x^2}$, $-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$

$\therefore \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} = 0$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\begin{pmatrix}\frac{\partial V}{\partial x}\\ \frac{\partial V}{\partial y}\end{pmatrix}\cdot\underbrace{\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix}}_{\text{これが } u(x,y)=C \text{ の曲線の法線方向を向く}} = \begin{pmatrix}\frac{\partial V}{\partial x}\\ \frac{\partial V}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial V}{\partial y}\\ -\frac{\partial V}{\partial x}\end{pmatrix} = 0 \quad (\text{V と u は直交})$$

Vは電位であり、電位と直交するものは、電気力線である。

$\omega = Az^{\frac{1}{2}} \longrightarrow \omega^2 = A^2 z$

$u^2 - V^2 + i\cdot 2uV = A^2(x+yi)$

$\begin{cases}u^2 - V^2 = A^2 x\\ 2uV = A^2 y\end{cases}$

$\therefore y^2 = \frac{4V^2}{A^4}(A^2 x + V^2)$

$V = V_0$のとき $y^2 = \frac{4V_0^2}{A^4}(A^2 x + V_0^2)$

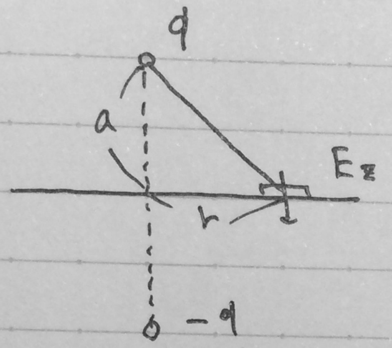
ここで $V_0 = 0$とすると

$y^2 = 0 \Longleftrightarrow y = 0 \quad (x \geqq 0)$

これらの考察により、$\omega = Az^{\frac{1}{2}}$ 型の複素速度ポテンシャルは、

$y=0$, $x>0$ に電荷負の金属板があるときの状態を表している。

(3)



$$E_z = \frac{2q}{4\pi\varepsilon_0 (x^2+y^2+a^2)} \cdot \frac{a}{\sqrt{x^2+y^2+a^2}}$$

また、ガウスの定理より

$$-E_z \cdot dx\,dy = \frac{\sigma(x,y)\,dx\,dy}{\varepsilon_0}$$

$$\sigma(x,y) = -\frac{q\ a}{2\pi(x^2+y^2+a^2)^{\frac{3}{2}}}$$